JOURNAL OF APPROXIMATION THEORY 75, 85–106 (1993)

Row Convergence of Algebraic Approximations

A. W. McINNES AND T. H. MARSHALL*

Department of Mathematics,
University of Canterbury, Christchurch, New Zealand

Communicated by Doron S. Lubinsky

Received February 18, 1992; accepted in revised form July 13, 1992

An **n** algebraic function of degree p satisfies an algebraic equation of degree p, whose polynomial coefficients have maximum degrees given by the vector **n**. If a function which is analytic at the origin is approximated by an **n** algebraic function of degree p, the table of approximations is a table of dimension $p+1$. Under suitable conditions, the sequence of algebraic approximations along an arbitrary "row" (a line parallel to an arbitrary axis in the table) converges to a given meromorphic function, unitformly on a suitable compact set.

1. INTRODUCTION

This paper discusses some convergence properties of the "rows" of the algebraic (Hermite–Padé) approximation. The results extend those of Baker and Lubinsky [3], which are, in turn, generalizations of the classical de Montessus theorem on the convergence of Padé approximants [1, 2].

Many of the ideas here are based on the work of Baker and Lubinsky [3], which also contains an extensive bibliography of previous investigations. However, one significant difference is that Baker and Lubinsky link the existence of an essentially unique algebraic form with the existence of a unique algebraic multiplier. Furthermore, these authors consider convergence only along "rows" parallel to the first axis of the table of algebraic approximations, and obtain only necessary conditions for the existence of a unique algebraic multiplier.

In this paper the concept of the existence of a unique algebraic multiplier has been decoupled from the concept of the existence of an essentially unique algebraic form, and consequently, of the algebraic approximation. It has been shown by McInnes [5] that an essentially unique algebraic form may be identified for any $f(z)$ which is analytic at the origin. The table of

* Present address: Department of Mathematics, University of Auckland, Auckland, New Zealand.

0021-9045/93 $5.00

n algebraic approximations of degree p is a table of dimension $p+1$. In this paper an "ith row" refers to a sequence of approximations along a line parallel to the ith component of the $(p+1)$-vector **n** (i.e., parallel to the ith axis in the table). One question considered here is to identify necessary and sufficient conditions under which an essentially unique i-multiplier (associated with the "ith row") exists for a given meromorphic function. The convergence theorem given in this paper is extended to consider convergence along an arbitrary "ith row" in the table of algebraic approximations. It is shown that the sequence of algebraic approximations converges to a given meromorphic function $f(z)$, uniformly on a suitable compact set.

In the remainder of this section the previous results are reviewed after establishing the basic definitions and notation. The main results are stated in Section 2 and proved in Section 3. Some comments and examples conclude the paper in Section 4.

Definitions and Notation

The **n** algebraic approximation of degree p can be defined as follows (McInnes [5]).

Let $f(z)$ be defined and analytic at $z=0$. Let p be a positive integer and let $n_0, n_1, ..., n_p$ be a set of integers all $\geqslant -1$. Choose a finite sequence of polynomials, $a_0(z), a_1(z), ..., a_p(z)$, not all zero, and of degrees not exceeding $n_0, n_1, ..., n_p$, respectively (where the polynomial of degree -1 is to be interpreted as the zero polynomial), such that

$$P(f, z) \equiv \sum_{j=0}^{p} a_j(z)\, f(z)^j = O(z^N), \tag{1.1}$$

where

$$N = \left[\sum_{j=0}^{p} (n_j + 1) \right] - 1.$$

The function $P(f, z)$ satisfying (1.1) is referred to as an *algebraic form* of the type $\mathbf{n} = (n_0, n_1, ..., n_p)$ and degree p.

Because Eq. (1.1) determines a homogeneous system of N linear equations in $N+1$ unknowns (viz., the coefficients of each $a_j(z)$), there will always exist a non-trivial solution, $(a_0(z), a_1(z), ..., a_p(z))$, to (1.1). When this solution space is one-dimensional, any two non-trivial solutions will be non-zero scalar multiples of each other. In this case the (non-trivial) solution is said to be *essentially unique*. A unique representative of this class of solutions may be identified by using a suitable normalization, such as requiring that the coefficients in the polynomials are no greater than one in absolute value with equality occurring in at least one case.

In general, the solution space for (1.1) may have more than one dimension. In this case we restore uniqueness by replacing N in (1.1) by $N+S$, where the "surplus" $S>0$ is chosen to be as large as possible (see [5, Thm. 3]). If $f(z)$ is an algebraic function of degree $p-k$, $k \geqslant 1$, then $S=\infty$ and the unique representative is chosen as the algebraic form of minimal degree, which will recover the algebraic function.

Given this unique algebraic form $P^*(f, z)$, it is clear that an algebraic function approximation $Q(z)$ may be defined by $P^*(Q, z)=0$. From the general theory of algebraic functions, it is known that this equation *normally* has p distinct analytic branches at the origin. This occurs when $\partial P^*(f, z)/\partial f|_{z=0} \neq 0$, and the **n** algebraic form $P^*(f, z)$ is called *normal* in this case [5]. (Normal and non-normal algebraic forms are discussed in McInnes [5]).

If the algebraic form $P^*(f, z)$ is *normal*, then the corresponding **n** *algebraic approximation of degree* p to $f(z)$ is defined as the unique solution, $Q(z)$, of

$$P^*(Q, z) \equiv \sum_{j=0}^{p} a_j(z)\, Q(z)^j = 0, \tag{1.2}$$

subject to the initial condition

$$Q(0)=f(0).$$

The case $p=1$ is the well known Padé approximation in which $Q(z)$ is rational. There exists an infinite sequence of Padé approximations along any "row" [2, Thm. 1.4.5], and the convergence of a sequence of these approximations is given by the de Montessus theorem [1; 2, Theorem 6.2.2].

The existence of an infinite sequence of rational approximations was extended to the existence of an infinite sequence of quadratic ($p=2$) approximations in [4]. The existence of arbitrary algebraic approximations of general degree has been subsequently shown in [5]. It remains to investigate the convergence properties of such a sequence.

Baker and Lubinsky [3] have shown that if

(a) $f(z)$ is analytic in the open disc $B(0, R)=\{z : |z|<R\}$, $(0<R \leqslant \infty)$ except for a finite number of poles (which exclude the origin),

(b) $K \subseteq B(0, R)$ is compact, simply connected, contains a neighborhood of the origin and excludes the poles of f and zeros of $\partial P_\infty/\partial f$, (refer to Theorem 2.3 for this notation),

(c) n_0, the nominal degree of $a_0(z)$, tends to infinity, while n_j remains fixed for $j>0$, then the approximations determined by (1.2) are uniquely defined on K for sufficiently large n_0, and converge, uniformly on K, to $f(z)$ as $n_0 \to \infty$.

In this paper, it is shown that a similar result holds if any of the $n_j \to \infty$, while the other n_i, $i \neq j$ remain fixed (convergence along an arbitrary "row"). Some of the hypotheses in [3] are dropped, including the simple connectivity of K. The notation used is modelled largely on that of [5], although there is also clearly a debt to the notation used in [3], with the notable exception that no special importance is attached to the $a_0(z)$ term.

2. Statement of Results

In order to prove the main theorem we need a preliminary result which is also of some independent interest. Roughly speaking, this theorem states that, where the equation $F(z, y) = 0$ determines a function $y = f(z)$ by the implicit function theorem, the function f depends continuously on the analytic function F. (Function spaces are given the compact-open topology).

THEOREM 2.1. *Let U and V be open in $\mathbb{C}$, $f : U \to V$ be analytic, and W be any open set in $\mathbb{C}^2$ containing $\{(z, f(z)) : z \in U\}$. Let F, F_L $(L = 1, 2, 3, \ldots)$ be analytic functions in two variables defined on W such that*

$$\lim_{L \to \infty} F_L(z, y) = F(z, y) \tag{2.1}$$

uniformly on compact subsets of W and

$$F(z, f(z)) = 0, \qquad z \in U. \tag{2.2}$$

Let $\mathcal{U}$ be an open subset of U whose closure in U is compact and suppose that

$$\partial F(z, y)/\partial y \,|_{y = f(z)} \neq 0 \qquad \text{for} \quad z \in \bar{\mathcal{U}}. \tag{2.3}$$

Then for all large enough L there are unique analytic functions $f_L : \mathcal{U} \to V$ such that $\{(z, f_L(z)) : z \in \mathcal{U}\} \subseteq W$ and such that

$$F_L(z, f_L(z)) = 0, \qquad z \in \mathcal{U}, \tag{2.4}$$

and

$$\lim_{L \to \infty} f_L(z) = f(z) \qquad \text{uniformly on } \mathcal{U}. \tag{2.5}$$

This result will be used to obtain uniform convergence of the sequence of algebraic approximations along a "row." However, we first need to generalize some of the definitions in [3].

It is assumed throughout that $f(z)$ is analytic except for isolated poles in some open disc $B(0, R)$, $(0 < R \leqslant \infty)$.

DEFINITION 2.2. Let p be a positive integer, i a non-negative integer not exceeding p and $\mathbf{n}^{(i)} = (n_0, n_1, ..., n_{i-1}, n_{i+1}, ..., n_p)$ be a p-vector of integers each $\geqslant -1$.

A p-vector, $\mathbf{a}^{(i)}(z) = (a_0(z), ..., a_{i-1}(z), a_{i+1}(z), ..., a_p(z))$, of polynomials is an *i-multiplier* of type $\mathbf{n}^{(i)}$ for $f(z)$ on $B(0, R)$ if

$$\begin{aligned} &\text{(i)} \quad \mathbf{a}^{(i)}(z) \not\equiv \mathbf{0}, \text{ and has degree not exceeding } \mathbf{n}^{(i)}, \\ &\text{(ii)} \quad \sum_{\substack{j=0 \\ j \neq i}}^{p} a_j(z) f(z)^{j-i} \text{ is analytic in } B(0, R). \end{aligned} \tag{2.6}$$

Note that if (2.6) is satisfied then it follows that $f(z)$ has only finitely many zeros in $B(0, R)$ (except when $i = 0$) and only finitely many poles in $B(0, R)$ (except when $i = p$). The 0-multiplier in this definition is the same as the algebraic multiplier defined in [3].

Given an i-multiplier $\mathbf{a}^{(i)}(z)$, we *define* the function $a_i(z)$ (analytic in $B(0, R)$) by

$$a_i(z) \equiv -\sum_{\substack{j=0 \\ j \neq i}}^{p} a_j(z) f(z)^{j-i}.$$

Note that in general $a_i(z)$ will *not* be a polynomial. Clearly the notion established in the above definition conflicts with the standard basic notation for the algebraic form in (1.1). The notation for the appropriate algebraic form is modified in an obvious way in the following theorem.

Since any non-zero constant multiple of an i-multiplier is also an i-multiplier, the i-multiplier will be called *essentially unique* [3] if any other i-multiplier with these properties has the form $c\mathbf{a}^{(i)}(z)$, where $c \neq 0$. For the remainder of this paper, "unique" should always be interpreted in the sense of "essentially unique," since, as in [5], a unique representative of this class of i-multipliers may be identified by choosing a suitable normalization of the vector of coefficients of the non-trivial vector $\mathbf{a}^{(i)}(z)$.

If the function $f(z)$ has q poles and r zeros (both counted with multiplicity) in $B(0, R)$, then (2.6) gives rise to a homogeneous system of $(p-i)q + ir$ linear equations in the coefficients of the polynomials $a_j(z)$, $j = 0, 1, ..., i-1, i+1, ..., p$. (In the Laurent expansions of the sum in (2.6) about each pole and zero of $f(z)$ the coefficients of negative powers must all be set to zero.) Note that r may be infinite when $i = 0$, and q may be infinite when $i=p$, but otherwise (2.6) implies that q, r are finite as observed above. If we define

$$N^{(i)}+1=\sum_{\substack{j=0\\ j\neq i}}^{p}(n_j+1), \tag{2.7}$$

then the number of unknown coefficients in this linear system is $N^{(i)}+1$ $(=N-n_i)$. An i-multiplier of type $\mathbf{n}^{(i)}$ for $f(z)$ must therefore exist when

$$N^{(i)}\geqslant(p-i)q+ir. \tag{2.8}$$

On the other hand, (essential) uniqueness of the multiplier means that the number of unknowns cannot exceed the number of equations by more than 1. That is,

$$N^{(i)}\leqslant(p-i)q+ir. \tag{2.9}$$

When the equality

$$N^{(i)}=(p-i)q+ir \tag{2.10}$$

holds, then the i-multiplier may be essentially unique, but, as Theorem 2.4 will show, not necessarily.

The main result is now stated.

THEOREM 2.3. *Let $f(z)$ be analytic at the origin and have q poles and r zeros (counted with multiplicity) in the open disc $B(0, R)$, $(0<R\leqslant\infty)$. Let p be a positive integer and i a non-negative integer satisfying $0\leqslant i\leqslant p$. Let $\mathbf{n}^{(i)}=(n_0, n_1, ..., n_{i-1}, n_{i+1}, ..., n_p)$ where the n_j are integers $\geqslant -1$, and let*

$$N^{(i)}=\left[\sum_{\substack{j=0\\ j\neq i}}^{p}(n_j+1)\right]-1=(p-i)q+ir.$$

(i) *For all $n_i\in\mathbb{Z}^+$, there is an essentially unique $\mathbf{n}=(n_0, n_1, ..., n_{i-1}, n_i, n_{i+1}, ..., n_p)$ algebraic form of degree p which is of maximal order, and which may be chosen uniquely by a suitable normalization of the vector of coefficients of the non-trivial vector of coefficient polynomials $\mathbf{a}(z)$. Denote this unique vector of polynomials by $\mathbf{a}_{n_i}(z)=(a_{0,n_i}(z), a_{1,n_i}(z), ..., a_{p,n_i}(z))$.*

(ii) *For every infinite sequence $\mathscr{S}$ of integers, there is a subsequence $\mathscr{S}'$ of $\mathscr{S}$, and an i-multiplier $\mathbf{a}^{(i)}(z)$ for $f(z)$ such that*

$$\lim_{\substack{n_i\to\infty\\ n_i\in\mathscr{S}'}} a_{j,n_i}(z)=a_j(z), \qquad j=0(1)p, \tag{2.11}$$

uniformly in compact subsets of $B(0, R)$ *for* $j = i$, *and uniformly in compact subsets of* $\mathbb{C}$ *for* $j \neq i$.

(iii) *Let* $\mathcal{S}, \mathcal{S}'$ *be as in* (ii) *and let*

$$P_\infty(f, z) \equiv \sum_{j=0}^{p} a_j(z) f(z)^j,$$

so that

$$\frac{\partial P_\infty(f, z)}{\partial f} = \sum_{j=1}^{p} j a_j(z) f(z)^{j-1}.$$

Let $\mathcal{U} \subseteq B(0, R)$ *be open and contain the origin. If* $\bar{\mathcal{U}}$ *excludes the poles of* $f(z)$ *and the zeros of* $\partial P_\infty(f, z)/\partial f$ (*in particular this entails that* $\partial P_\infty(f, z)/\partial f|_{z=0} \neq 0$), *then, for sufficiently large* n_i, *the algebraic approximation* $Q_{n_i}(z)$ *defined by*

$$P_{n_i}(Q_{n_i}, z) \equiv \sum_{j=0}^{p} a_{j,n_i}(z)\, Q_{n_i}(z)^j = 0,$$

subject to the initial condition

$$Q_{n_i}(0) = f(0),$$

is defined and analytic at the origin, and can be extended to a (*single-valued*) *function on* $\mathcal{U}$, *and we have*

$$\lim_{\substack{n_i \to \infty \\ n_i \in \mathcal{S}'}} Q_{n_i}(z) = f(z), \qquad \text{uniformly for } z \in \mathcal{U}. \tag{2.12}$$

(iv) *If* $f(z)$ *has an* essentially unique *i-multiplier of type* $\mathbf{n}^{(i)}$ *then the limits in* (2.11) *and* (2.12) *are valid with* $\mathcal{S}' = \{1, 2, 3, \ldots\}$.

In order to apply the previous theorem it is of interest to know for which $\mathbf{n}^{(i)}$ satisfying equation (2.10), there is a unique i-multiplier of type $\mathbf{n}^{(i)}$ for $f(z)$ on $B(0, R)$, for *all* $f(z)$ with q poles and r zeros in $B(0, R)$.

Suppose that, for such an $\mathbf{n}^{(i)}$ and $f(z)$, $\mathbf{a}^{(i)}(z)$ is a unique i-multiplier of type $\mathbf{n}^{(i)}$ as defined in Definition 2.2. Let

$$\mathbf{n}_1^{(i)} = (n_{i-1}, n_{i-2}, \ldots, n_1, n_0)$$

and

$$\mathbf{n}_2^{(i)} = (n_{i+1}, n_{i+2}, \ldots, n_p).$$

Then we have

$$\sum_{j=0}^{i-1} a_j(z) f(z)^{j-i} + \sum_{j=i+1}^{p} a_j(z) f(z)^{j-i} \tag{2.13}$$

is analytic in $B(0, R)$. Since the two sums in Eq. (2.13) can have poles only at the zeros and poles of $f(z)$ respectively, they must each be analytic in $B(0, R)$.

In view of the uniqueness of $\mathbf{a}^{(i)}(z)$ this condition can be satisfied only by the two possibilities:

(i) $\mathbf{b}_1(z) = (a_{i-1}(z), a_{i-2}(z), ..., a_0(z))$ is the unique 0-multiplier of type $\mathbf{n}_1^{(i)}$ for $[f(z)]^{-1}$ and $\mathbf{b}_2(z) = (a_{i+1}(z), a_{i+2}(z), ..., a_p(z)) \equiv \mathbf{0}$.

(ii) $\mathbf{b}_1(z) \equiv \mathbf{0}$ and $\mathbf{b}_2(z)$ is the unique 0-multiplier of type $\mathbf{n}_2^{(i)}$ for $f(z)$.

Suppose (i) holds. Then an inequality of the type (2.9) must hold for this case. That is, the 0-multiplier of type $\mathbf{n}_1^{(i)}$ for $[f(z)]^{-1}$ satisfies

$$N_1^{(i)} = \left[\sum_{j=0}^{i-1} (n_j + 1)\right] - 1 \leqslant (i-0)r + 0q = ir. \tag{2.14}$$

Since $\mathbf{b}_2(z) \equiv \mathbf{0}$, there is no multiplier of type $\mathbf{n}_2^{(i)}$, and hence an inequality of the type (2.8) cannot hold in this case. That is,

$$N_2^{(i)} = \left[\sum_{j=i+1}^{p} (n_j + 1)\right] - 1 < (p-i-0)q + 0r = (p-i)q,$$

and hence

$$N_2^{(i)} \leqslant (p-i)q - 1. \tag{2.15}$$

From the definition (2.7),

$$\begin{aligned} N^{(i)} &= N_1^{(i)} + N_2^{(i)} + 1 \\ &\leqslant ir + (p-i)q - 1 + 1, \end{aligned}$$

using (2.14), (2.15),

$$= (p-i)q + ir.$$

But by hypothesis, (2.10) holds, and the previous inequality must be an equality. Hence equality holds also in (2.14), (2.15).

A similar argument for the case (ii) gives $N_1^{(i)} = ir - 1$ and $N_2^{(i)} = (p-i)q$ for the second case.

The problem of finding $\mathbf{n}^{(i)}$ satisfying (2.10) for which a unique i-multiplier always exists thus reduces to the case $i=0$, along with a closely related problem of finding $\mathbf{n}^{(i)}$ for which 0-multipliers never exist. The following theorem characterizes these $\mathbf{n}^{(i)}$.

THEOREM 2.4. *Let $q \geqslant 0$.*

(i) *If $\mathbf{n}^{(0)} = (n_1, ..., n_p)$ satisfies*

$$N^{(0)} = \left[\sum_{j=1}^{p} (n_j + 1) \right] - 1 = pq \tag{2.16}$$

then there is a unique 0-multiplier of type $\mathbf{n}^{(0)}$ in $B(0, R)$, for every $0 < R \leqslant \infty$ and $f(z)$ with q poles in $B(0, R)$, if and only if

$$\mathbf{n}^{(0)} = (\mathbf{m}_1; \mathbf{m}_2; ...; \mathbf{m}_J) \tag{2.17}$$

where, in (2.17), all but one of the $\mathbf{m}_j$, $j = 0(1)J$, are of the form

$$\mathbf{m}_j = (\underbrace{-1, -1, ..., -1}_{s-1}, sq - 1), \qquad s > 0, \tag{2.18}$$

and the remaining one of the $\mathbf{m}_j$ is either of the form

$$\mathbf{m}_j = (\underbrace{-1, ..., -1}_{s-1}, sq), \qquad s > 0, \tag{2.19}$$

or

$$\mathbf{m}_j = (\underbrace{-1, ..., -1, 0, -1, ..., -1}_{s-1}, sq - 1), \qquad s > 0, \tag{2.20}$$

where, in (2.20), the zero may be any one of the first $s - 1$ entries.

(ii) *If $\mathbf{n}^{(0)}$ satisfies $N^{(0)} = pq - 1$ then there is no 0-multiplier of type $\mathbf{n}^{(0)}$ in $B(0, R)$ for any $0 < R \leqslant \infty$ and any $f(z)$ with q poles in $B(0, R)$, if and only if*

$$\mathbf{n}^{(0)} = (\mathbf{m}_1; \mathbf{m}_2; ...; \mathbf{m}_J) \tag{2.21}$$

where in (2.21) every $\mathbf{m}_j$ is of the form given by (2.18).

Remarks. 1. Equation (2.16) is (2.10) for $i = 0$.

2. Case (i) is the situation where the number of unknowns (the $N^{(0)} + 1$ unknown polynomial coefficients) is one more than the number, pq, of linear equations arising from (2.6), and we might expect an essen-

tially unique non-trivial solution. The theorem identifies the types $\mathbf{n}^{(0)}$ for which this unique solution actually occurs.

Case (ii) is the situation where the number of unknown polynomial coefficients and the number of linear equations arising from (2.6) are equal, and we might expect only the trivial solution (i.e., there is no multiplier). The theorem identifies the types $\mathbf{n}^{(0)}$ for which there is no multiplier.

3. Proofs of the Results

Proof of Theorem 2.1. Let

$$G(t)=\{(z, y) \mid z \in \bar{\mathscr{U}}, |y-f(z)| \leqslant t\} \qquad \text{for} \quad t \geqslant 0.$$

Using (2.3), there exists $h>0$ for which

$$G(h) \subseteq W,$$

and

$$c=\inf_{(z, y) \in G(h)} (|\partial F(z, y)/\partial y|)>0. \tag{3.1}$$

Let

$$d=\sup_{(z, y) \in G(h)} (|\partial^2 F(z, y)/\partial y^2|). \tag{3.2}$$

Using the Taylor series expansion of $F(z, y)$, about $y=f(z)$, gives for $z \in \bar{\mathscr{U}}$, and $|y-f(z)| \leqslant h$,

$$F(z, y)=F(z, f(z))+(y-f(z))\, \partial F(z, f(z))/\partial y+\bar{R},$$

where, by (3.2),

$$|\bar{R}| \leqslant \tfrac{1}{2} d\, |y-f(z)|^2.$$

Using (2.2) and (3.1), this gives

$$|F(z, y)| \geqslant |y-f(z)|\, (c-\tfrac{1}{2} d\, |y-f(z)|). \tag{3.3}$$

Hence, for $|y-f(z)| \leqslant \delta \leqslant \delta_0=\min(h, cd^{-1})$ we have

$$|F(z, y)| \geqslant \tfrac{1}{2} c\, |y-f(z)|. \tag{3.4}$$

Using (2.1) and (3.1), we may choose $L_0=L_0(\delta)$ large enough so that for all $L \geqslant L_0$,

$$|F(z, y)-F_L(z, y)|<\tfrac{1}{2} c \delta, \tag{3.5}$$

and

$$|\partial F_L(z, y)/\partial y| > 0, \tag{3.6}$$

on $G(\delta)$.

By (2.2) and (2.3), $y = f(z)$ is a simple zero of $F(z, y) = 0$ and, by (3.4), there are no other zeros in $B(f(z), \delta)$. Thus using (3.5), Rouché's Theorem now applies and gives, for each $L \geqslant L_0$ and $z \in \bar{\mathscr{U}}$ a unique solution, $y = f_L(z)$, of

$$F_L(z, y) = 0, \tag{3.7}$$

for which

$$|y - f(z)| < \delta. \tag{3.8}$$

It is immediate from (3.7) that f_L has the required property (2.4). Since δ may be chosen arbitrarily small, (3.8) shows that f_L also satisfies (2.5).

It remains only to show that $f_L(z)$ is an analytic function on $\mathscr{U}$.

In view of (3.6) and (3.7) with $\delta = \delta_0$, the implicit function theorem gives, for any $L \geqslant L_0(\delta_0)$ and $z_0 \in \mathscr{U}$, an analytic function g_L defined in some neighborhood, B_0, of z_0, for which

$$g_L(z_0) = f_L(z_0) \tag{3.9}$$

and

$$F_L(z, g_L(z)) = 0 \qquad \text{for} \quad z \in B_0. \tag{3.10}$$

By (3.9) and the continuity of g_L and f, we may assume B_0 chosen so small that

$$|g_L(z) - f(z)| < \delta_0, \qquad z \in B_0.$$

Then, by the uniqueness of the solution of (3.7), (3.8) we have

$$g_L(z) = f_L(z) \qquad \text{on } B_0.$$

Thus f_L is analytic at $z = z_0$, and, since the choice of z_0 is arbitrary, this equality holds on all of $\mathscr{U}$. ∎

The proof of Theorem 2.3 is modelled closely on the proofs of the analogous results given by Baker and Lubinsky [3].

Proof of Theorem 2.3(i). The existence and uniqueness of the $\mathbf{n}$ algebraic form of degree p has been shown by McInnes [5]. Hence, given $\mathbf{n}^{(i)}$, there exists a sequence of unique, non-trivial vectors of polynomials, $\mathbf{a}_{n_i}(z)$. ∎

Proof of Theorem 2.3(ii). Let $\mathbf{a}_{n_i}^{(i)}(z)$ be the p-vector of polynomials which consists of the vector $\mathbf{a}_{n_i}(z)$ with the term $a_{i,n_i}(z)$ missing. That is,

$$\mathbf{a}_{n_i}^{(i)}(z) = (a_{0,n_i}(z), a_{1,n_i}(z), ..., a_{i-1,n_i}(z), a_{i+1,n_i}(z), ..., a_{p,n_i}(z)).$$

To begin, we show that

$$\mathbf{a}_{n_i}^{(i)}(z) \not\equiv \mathbf{0}, \qquad n_i = 0, 1, 2, ... \tag{3.11}$$

Assume on the contrary that $\mathbf{a}_{n_i}^{(i)}(z) \equiv \mathbf{0}$. Since the $\mathbf{n}$ algebraic form of degree p is $O(z^N)$, this algebraic form reduces to

$$a_{i,n_i}(z)\, f(z)^i = O(z^N). \tag{3.12}$$

Since $\mathbf{a}_{n_i}(z)$ is a non-trivial vector, $a_{i,n_i}(z) \not\equiv 0$, and the left side of (3.12) has a zero or order at most $ir + n_i$. But by (2.10) (a hypothesis of this theorem),

$$ir + n_i < N^{(i)} + n_i + 1 \qquad (\text{since } (p-i)q > -1).$$

Further $N^{(i)} + n_i + 1 = N$, and so this inequality contradicts (3.12). Hence (3.11) holds.

Now normalize the vector $\mathbf{a}_{n_i}(z)$ so that the coefficients of each polynomial in $\mathbf{a}_{n_i}^{(i)}(z)$ have absolute value at most one with equality for at least one coefficient. A standard diagonal argument allows us to choose a subsequence $\mathscr{S}'$, of the sequence $\mathscr{S}$ of integers n_i, in which each coefficient in each polynomial of $\mathbf{a}_{n_i}^{(i)}(z)$ converges.

If we now define $a_j(z)$ by letting each of its coefficients be the limit, as $n_i \to \infty$ through $\mathscr{S}'$, of the corresponding coefficients in $a_{j,n_i}(z)$, it is clear that (2.11) holds for $j \neq i$.

To complete the proof we must show that

$$\mathbf{a}^{(i)}(z) = (a_0(z), ..., a_{i-1}(z), a_{i+1}(z), ..., a_p(z))$$

is an i-multiplier and that (2.11) holds for $j = i$.

Let

$$\begin{aligned} S(z) &= \prod_{j=1}^{l} (z - z_j)^{m_j}, \\ U(z) &= \prod_{j=1}^{l'} (z - z_j')^{m_j'}, \end{aligned} \tag{3.13}$$

where $z_1, ..., z_l$ $(z_1', ..., z_{l'}')$ are the distinct poles (zeros) of $f(z)$ with respective multiplicities $m_1, ..., m_l$ $(m_1', ..., m_{l'}')$.

Let

$$T(z) = S(z)^{p-i}\, U(z)^i,$$

$$P_{n_i}(f, z) = \sum_{j=0}^{p} a_{j,n_i}(z)\, f(z)^j,$$

$$P_{n_i}^{(i)}(f, z) = \sum_{\substack{j=0\\ j\neq i}}^{p} a_{j,n_i}(z)\, f(z)^j.$$

Then

$$T(z)\, P_{n_i}(f, z)/(f(z)^i z^N) = T(z) \sum_{j=0}^{p} a_{j,n_i}(z)\, f(z)^{j-i}/z^N,$$

which is analytic in $B(0, R)$. By Cauchy's integral formula with $|z| < \rho < R$, we have

$$\begin{aligned} &T(z)\, P_{n_i}(f, z)/(f(z)^i z^N)\\ &\quad = \frac{1}{2\pi i}\int_{|t|=\rho} T(t)\, P_{n_i}(f, t)/(f(t)^i\, t^N(t-z))\, dt,\\ &\quad = \frac{1}{2\pi i}\left[\int_{|t|=\rho} T(t)\, P_{n_i}^{(i)}(f, t)/(f(t)^i\, t^N(t-z))\, dt\right.\\ &\qquad \left. + \int_{|t|=\rho} T(t)\, a_{i,n_i}(t)/(t^N(t-z))\, dt\right]. \end{aligned} \tag{3.14}$$

Since, by (2.10), $T(t)$ is of degree $N^{(i)} = N - n_i - 1$, the integrand of the second integral in (3.14) is $O(t^{-2})$ as $|t| \to \infty$, is analytic as a function of t in $|t| \geq \rho$, and hence (letting $\rho \to \infty$) this integral vanishes.

In the first integral in (3.14), the terms $T(t)\, f(t)^{j-i}$ are analytic and hence bounded, and the normalization of the $\mathbf{a}_{n_i}^{(i)}(z)$ ensures that the polynomial coefficients remain bounded as $n_i \to \infty$. Thus, since $\rho^{-N} = O(\rho^{-n_i})$ as $n_i \to \infty$, the first integral in (3.14) is $O(\rho^{-n_i})$ as $n_i \to \infty$, so that, since $\rho < R$ was arbitrary, we have, uniformly for $|z| \leq \rho'(<\rho) < R$,

$$\limsup_{n_i \to \infty} |T(z)\, P_{n_i}(f, z)/f(z)^i|^{1/n_i} \leq \frac{\rho'}{R}. \tag{3.15}$$

Together with (2.11) for $j \neq i$, this shows that, uniformly on compact subsets of $B(0, R)$, excluding poles and zeros of $f(z)$, we have

$$\lim_{\substack{n_i \to \infty\\ n_i \in \mathscr{S}'}} a_{i,n_i}(z) = -\sum_{\substack{j=0\\ j\neq i}}^{p} a_j(z)\, f(z)^{j-i} = a_i(z). \tag{3.16}$$

But since $\{a_{i,n_i}(z)\}_{n_i \in \mathscr{S}'}$ is a sequence of polynomials which converges uniformly on small circles centered at the poles and zeros of $f(z)$, it converges uniformly throughout the interior of these circles as well, and hence on arbitrary compact subsets of $B(0, R)$.

The left side of (3.16) is thus analytic in $B(0, R)$ and hence $\mathbf{a}^{(i)}(z)$ is an i-multiplier, and (2.11) holds for $j = i$. ∎

Proof of Theorem 2.3(iii). For $n_i \in \mathscr{S}'$, apply Theorem 2.1 with

$$P_{n_i}(y, z) = \sum_{j=0}^{p} a_{j,n_i}(z)\, y^j \qquad \text{in place of } F_L(z, y),$$

$$P_\infty(y, z) = \sum_{j=0}^{p} a_j(z)\, y^j \qquad \text{in place of } F(z, y),$$

$$Q_{n_i}(z) \qquad \text{in place of } f_L(z),$$

$$U = B(0, R) \backslash \{z \mid z \text{ is a pole of } f \text{ or a zero of } \partial P_\infty / \partial f\},$$

$$V = \mathbb{C}.$$

The uniqueness of the solution in Theorem 2.1 ensures that the $Q_{n_i}(z)$ given by this Theorem satisfies the initial condition $Q_{n_i}(0) = f(0)$. Note that since $\partial P_\infty(y, z)/\partial y|_{z=0} \neq 0$, for n_i sufficiently large, $P_{n_i}(f, z)$ is a normal algebraic form. ∎

Proof of Theorem 2.3(iv). The convergence of (2.11) and (2.12) when $\mathscr{S}' = \mathbb{Z}^+$ follows easily from parts (ii) and (iii), respectively, and the assumed uniqueness of the i-multiplier $\mathbf{a}^{(i)}(z)$. Note that by part (i) the sequence $\mathbf{a}_{n_i}(z)$ may be chosen uniquely. ∎

As a preliminary to proving Theorem 2.4 two simple lemmas are established.

LEMMA 3.1. *The vector* $\mathbf{a}^{(0)}(z) = (a_1(z), ..., a_p(z))$ *is a* 0*-multiplier of type* $\mathbf{n}^{(0)} = (n_1, ..., n_p)$ *for* $f(z)$ *in* $B(0, R)$ (*or* $\mathbf{a}^{(0)}(z) \equiv \mathbf{0}$) *if and only if, for each* k, $k = 1(1)p$, *either* $\mathbf{b}_k(z) = (a_k(z), ..., a_p(z))$ *is also a* 0*-multiplier of type* $(n_k, ..., n_p)$ *for* $f(z)$ *in* $B(0, R)$ *or* $\mathbf{b}_k(z) \equiv \mathbf{0}$.

Proof. If each $\mathbf{b}_k(z)$ is a 0-multiplier then in particular $\mathbf{b}_1(z) = \mathbf{a}^{(0)}(z)$ is a 0-multiplier.

Conversely, suppose that $\mathbf{a}^{(0)}(z)$ is a 0-multiplier for $f(z)$. For $k = 1(1)p$,

$$\sum_{j=k}^{p} a_j(z)\, f(z)^j$$

can have poles only at the poles of $f(z)$ and, in order to cancel the poles in the sum of the first $k-1$ terms of $\sum_{j=1}^{p} a_j(z) f(z)^j$, these poles can have order no greater than the corresponding poles of $f(z)^{k-1}$. Thus

$$\left[\sum_{j=k}^{p} a_j(z) f(z)^j\right] f(z)^{-(k-1)} = \sum_{j=k}^{p} a_j(z) f(z)^{j-k+1}$$

is analytic in $B(0, R)$. That is, $\mathbf{b}_k(z) = (a_k(z), ..., a_p(z))$ is the zero vector or is a 0-multiplier. ∎

LEMMA 3.2. *Let $\mathbf{n}^{(0)} = (n_1, ..., n_p)$. Then there is a 0-multiplier of type $\mathbf{n}^{(0)}$ for some $f(z)$ with q poles in some $B(0, R)$ if either of the following hold:*

(i) *For any k satisfying $1 \leqslant k \leqslant p$,*

$$\sum_{j=1}^{k} n_j \geqslant 1 + k(q-1). \tag{3.17}$$

(ii) *There are positive integers $v, w \leqslant p$ such that for some $m > 0$, $0 < v - w \leqslant m$ and*

$$n_v \geqslant mq, \tag{3.18}$$

and

$$n_w \geqslant 0. \tag{3.19}$$

The multiplier is not unique if any of the following hold:

(iii) *The inequality (3.17) is strict.*

(iv) *The conditions (3.18), (3.19) apply to two or more distinct pairs v, w.*

(v) *Both conditions (3.18), (3.19) hold strictly.*

(vi) *Conditions (3.18), (3.19) apply and (3.17) holds for some $k < v$.*

Proof. By inequalities (2.8) and (2.9) with $i = 0$ and $p = k$, the (strict) inequality (3.17) guarantees the existence (non-uniqueness) of a 0-multiplier of type $(n_1, ..., n_k)$ for any $f(z)$ with q poles in $B(0, R)$. These multipliers can be converted to 0-multipliers of type $\mathbf{n}^{(0)}$ by appending zeros to make the k-vector into a p-vector. This proves (i) and (iii).

Choose $R > 0$, distinct values $z_1, ..., z_l$ in $B(0, R)$ and integers $m_1, ..., m_l$ such that $\sum_{j=1}^{l} m_j = q$. Define $S(z)$ as in (3.13) and let $f(z) = S(z)^{-1}$. Then if (3.18), (3.19) hold, the vector $\mathbf{a}^{(0)}(z) = (a_1(z), ..., a_p(z))$ defined by letting

$$a_v(z) = S(z)^{v-w}, \qquad a_w(z) = -1, \qquad a_k(z) \equiv 0 \qquad \text{for} \quad k \neq v, w,$$

is a 0-multiplier of type $\mathbf{n}^{(0)}$. Moreover each different pair v, w gives a different multipler. This proves (ii) and (iv).

If each of (3.18), (3.19) hold strictly then each member of $\mathbf{a}^{(0)}(z)$ defined above may be multiplied by a linear function $\lambda(z) \not\equiv 0$ to give a new 0-multiplier which is still of type $\mathbf{n}^{(0)}$. Since the choice of $\lambda(z)$ is arbitrary, (v) follows.

Finally, when the conditions of (vi) hold, the multipliers that arise from (i) and (ii) are clearly distinct. ∎

Proof of Theorem 2.4. Let $\mathbf{n}^{(0)} = (n_1, ..., n_p)$ satisfying (2.16) be such that there is a unique 0-multiplier of type $\mathbf{n}^{(0)}$ for each $f(z)$ with q poles. We show that $\mathbf{n}^{(0)}$ must be of the form (2.17).

The case $q = 0$ is trivial, so assume $q \geqslant 1$. Let

$$v_j = c_j q + d_j, \qquad j = 1(1)J, \qquad c_j \geqslant 0, \qquad 0 \leqslant d_j < q,$$

be the positive entries of $\mathbf{n}^{(0)}$ listed from left to right. By Lemma 3.2(iii), each v_j has at least $c_j - 1$ entries to the left of it, by (v), none of these can be positive and, by (iv), at most one of the c_j entries preceding v_j can differ from -1. That is, v_j must be the rightmost member of a block $\mathbf{m}_j$ of the form either

$$\mathbf{m}_j = (\underbrace{-1, ..., -1}_{c_j}, v_j) \tag{3.20}$$

or

$$\mathbf{m}_j = (\underbrace{-1, ..., 0, ..., -1}_{c_j}, v_j), \tag{3.21}$$

where, in (3.21), the 0 can be any of the first c_j entries, or

$$\mathbf{m}_j = (\underbrace{-1, ..., -1}_{c_j - 1}, v_j), \tag{3.22}$$

where, in (3.22), the first entry in $\mathbf{m}_j$ is either n_1 or is immediately preceded by a positive entry. In these cases Lemma 3.2(iii) and (v), respectively, ensure that $d_j = 0$.

In addition to the blocks $\mathbf{m}_j$, $j = 1(1)J$, $\mathbf{n}^{(0)}$ may contain $D \geqslant 0$ individual entries of -1 and 0.

By Lemma 3.2(iii), (vi), at most one of the $\mathbf{m}_j$ can be of the form (3.21) or (3.22). If the exceptional block is of the form (3.21), then we have

$$p = \sum_{j=1}^{J} (c_j + 1) + D \geqslant \sum_{j=1}^{J} (c_j + 1), \tag{3.23}$$

with equality if and only if $D=0$, and

$$\sum_{j=1}^{p} n_j \leqslant 1+\sum_{j=1}^{J}(c_j q+d_j-c_j) \leqslant 1+(q-1)\sum_{j=1}^{J}(c_j+1), \tag{3.24}$$

with equality holding only if each $d_j=q-1$. By (2.16), $\sum_{j=1}^{p} n_j = 1+p(q-1)$, and combining the inequalities (3.23) and (3.24) gives $\sum_{j=1}^{p} n_j \leqslant 1+(q-1)p$. Hence, by (2.16), equality must hold in both (3.23) and (3.24), so $D=0$ and each $d_j=q-1$. On the other hand there must be at least one exceptional block else inequalities similar to (3.23) and (3.24) give $\sum_{j=1}^{p} n_j \leqslant (q-1)p$, which is contrary to (2.16). Hence $\mathbf{n}^{(0)}$ has the form given by (2.17).

If the exceptional block is of the type (3.22) a similar argument applies. In this case the right sides of (3.23) and (3.24) are reduced by 1 and $q-1$ respectively (the latter following because some $d_j=0$). Again using (2.16), it may be concluded that $\mathbf{n}^{(0)}$ has the form (2.17). A similar but simpler argument proves that in Theorem 2.4(ii), $\mathbf{n}^{(0)}$ must take the form (2.21).

To show sufficiency in part (i) (respectively part (ii)), let $\mathbf{n}^{(0)}$ take the form (2.17) (respectively (2.21)). We show that there is a unique 0-multiplier for every (respectively no 0-multiplier for any) $f(z)$ with q poles. The existence of a 0-multiplier follows from (2.16). The proof of uniqueness is by induction on the number of blocks in $\mathbf{n}^{(0)}$. Both parts (i) and (ii) are treated simultaneously.

For the induction step, suppose that Theorem 2.4 holds when this number is $J-1$, and let $\mathbf{n}^{(0)}$, of the form (2.17), have J blocks. Then we may write $\mathbf{n}^{(0)}=(\mathbf{m}_1;\mathbf{n}_{J-1})$ where $\mathbf{m}_1$ is a vector of length s and $\mathbf{n}_{J-1}=(\mathbf{m}_2;\ldots;\mathbf{m}_J)$. There are three possibilities:

(i) $\mathbf{n}_{J-1}$ is of the form (2.21), $\mathbf{m}_1$ (of length s) is of the form (2.19). Let $\mathbf{a}^{(0)}=(a_1(z),\ldots,a_p(z))$ be a 0-multiplier of type $\mathbf{n}^{(0)}$ for some $f(z)$ with q poles. By Lemma 3.1, $\mathbf{b}_{s+1}=(a_{s+1}(z),\ldots,a_p(z))$ of type $\mathbf{n}_{J-1}$ is either a 0-multiplier of type $\mathbf{n}_{J-1}$ or is identically zero. By the induction hypothesis the latter occurs, and since also, the first $s-1$ entries of $\mathbf{m}_1$ are -1, the sum (2.6) (with $i=0$) reduces to one term and we have $a_s(z)\,f(z)^s$ is analytic in $B(0,R)$, where $a_s(z)$ is of degree not exceeding $n_s=sq$. Clearly $a_s(z)$ is uniquely determined as $S(z)^s$, where $S(z)$ is defined in (3.13).

(ii) $\mathbf{n}_{J-1}$ is of the form (2.21), $\mathbf{m}_1$ is of the form (2.20). Arguing as in the first case we conclude that for some $w<s$,

$$a_w(z)\,f(z)^w+a_s(z)\,f(z)^s \tag{3.25}$$

is analytic in $B(0,R)$, where the degrees of $a_w(z)$ and $a_s(z)$ do not exceed 0 and $sq-1$, respectively. Since $a_w(z)\equiv 0$ implies $a_s(z)\equiv 0$, we may assume $a_w(z)=1$. Factorizing (3.25) gives

$$(1+a_s(z)\,f(z)^{s-w})\,f(z)^w. \tag{3.26}$$

The left factor of (3.26), and hence the function $g(z)=a_s(z)/S(z)^{s-w}$, must be analytic in $B(0, R)$. Hence at each pole, z_i, of $f(z)$ we require

$$1+g(z)(S(z)\,f(z))^{s-w}$$

to have a zero of multiplicity at least $m_i w$. That is, at each $z=z_i$

$$g(z)=-(S(z)\,f(z))^{w-s} \tag{3.27}$$

and

$$g^{(k)}(z)=0, \qquad 1\leqslant k\leqslant m_i w-1. \tag{3.28}$$

Since the degree of $g(z)$ does not exceed $sq-1-(s-w)q=wq-1$, these wq collocation conditions uniquely determine $g(z)$, and hence $a_s(z)$, and hence $\mathbf{a}^{(0)}(z)$.

(iii) $\mathbf{n}_{J-1}$ is of the form (2.17), $\mathbf{m}_1$ is of the form (2.18). In this case Lemma 3.1 and the induction hypothesis imply that $(a_{s+1}(z), ..., a_p(z))$ is the unique 0-multiplier of type $\mathbf{n}_{J-1}$ for $f(z)$, so that

$$h(z)=\sum_{j=s+1}^{p} a_j(z)\,f(z)^{j-s} \tag{3.29}$$

and hence

$$a_s(z)\,f(z)^s+f(z)^s\,h(z) \tag{3.30}$$

are analytic in $B(0, R)$. Factoring (3.30), a similar but easier argument to that used in case (ii) shows that $a_s(z)$, and hence $\mathbf{a}^{(0)}(z)$ is uniquely determined.

This completes the induction when $\mathbf{n}$ is of the form (2.17). When $\mathbf{n}^{(0)}$ is of the form (2.21) the induction step for part (ii) of the Theorem is similar to case (i) already considered.

The above arguments are easily adapted to prove the induction basis (i.e., $J=1$, so that $\mathbf{n}^{(0)}=\mathbf{m}_1$ and $\mathbf{n}_{J-1}$ is empty). ∎

4. Remarks and Examples

1. Baker and Graves-Morris [2] remark on a valuable duality property of Padé approximations. This duality property [1, Thm. 9.2; 2, Thm. 1.5.1] may be extended to algebraic forms of arbitrary degree.

Let $f(z)$ be defined and analytic in a neighborhood of $z=0$, and assume $f(0)\neq 0$. The $\mathbf{n}$ algebraic form of degree p is given by (1.1):

$$P(f, z)\equiv\sum_{j=0}^{p} a_j(z)\,f(z)^j=O(z^N)$$

Dividing by $f(z)^p$ gives

$$\sum_{j=0}^{p} a_{p-j}(z)(f(z)^{-1})^j = O(z^N),$$

which is an $\bar{\mathbf{n}} = (n_p, n_{p-1}, ..., n_0)$ algebraic form of degree p for $f(z)^{-1}$.

Moreover, if the $\mathbf{n}$ algebraic approximation of degree p to $f(z)$ is defined as the function $Q(z)$ satisfying

$$\sum_{j=0}^{p} a_j(z)\,Q(z)^j = 0, \qquad \text{subject to} \quad Q(0) = f(0),$$

then

$$\begin{aligned}\sum_{j=0}^{p} a_{p-j}(z)\,Q(z)^{-j} &= Q(z)^{-p}\left\{\sum_{j=0}^{p} a_{p-j}(z)\,Q(z)^{p-j}\right\}\\ &= Q(z)^{-p}\{0\} = 0,\end{aligned}$$

and conversely.

Clearly $Q(0) = f(0)$ iff $Q(0)^{-1} = f(0)^{-1}$. Hence it may be concluded that if $P(f, z)$ is an $\mathbf{n}$ normal algebraic form of degree p for $f(z)$, with $f(0) \neq 0$, and a vector of coefficient polynomials

$$\mathbf{a}(z) = (a_0(z), a_1(z), ..., a_p(z)),$$

then the $\bar{\mathbf{n}}$ algebraic form of degree p for $f(z)^{-1}$ has the vector of coefficient polynomials

$$\bar{\mathbf{a}}(z) = (a_p(z), a_{p-1}(z), ..., a_0(z)).$$

Furthermore, the $\bar{\mathbf{n}}$ algebraic approximation of degree p to $f(z)^{-1}$ may be defined as the unique solution, $\bar{Q}(z) = Q(z)^{-1}$, to

$$\sum_{j=0}^{p} a_{p-j}(z)\,\bar{Q}(z)^j = 0,$$

subject to the initial condition

$$\bar{Q}(0) = Q(0)^{-1} = f(0)^{-1}.$$

2. In the case of rational functions, $p = 1$, the inequality (2.8) reduces to $N^{(0)} \geqslant q$ and $N^{(1)} \geqslant r$ where $i = 0$ and $i = 1$, respectively. That is, $n_1 \geqslant q$ and $n_0 \geqslant r$ for $i = 0, 1$ respectively. When equality holds there is a unique 0-multiplier $(S(z))$ of type $\mathbf{n}^{(0)} = (n_1)$ and a unique 1-multiplier $(U(z))$ of type $\mathbf{n}^{(1)} = (n_0)$, where $S(z)$ and $U(z)$ are defined in (3.13). Applying

Theorem 2.3(iv) in these cases gives, respectively, the classical de Montessus theorem for convergence of Padé approximants (for example, [1, Thm. 11.1; 2, Thm. 6.2.2]) and its dual (for example, [1, Corol. 11.3]).

3. Even in the case $p=1$, $i=0$, convergence of the whole sequence of approximations generally fails in the absence of a unique i-multiplier (for example, [2, p. 238; 1, p. 147]).

4. In this paper we have chosen $P_{n_i}(f, z)$ to be essentially unique for all n_i. However, this choice is not essential, since any representative in the solution space of (1.1) may be used in these proofs. The argument in [3, Thm. 2.5(i)] can be adapted to show that in the presence of a unique algebraic multiplier the problem of non-uniqueness of $P_{n_i}(f, z)$ for sufficiently large n_i, does not arise.

5. The following simple example illustrates Theorem 2.3.

EXAMPLE. Let

$$f(z)=\frac{e^z(1-3z)}{(1-z)(1-2z)}.$$

Set $R>1$. Thus $q=2$, $r=1$. Consider the sequence of $\mathbf{n}=(0, n_1, 2)$ quadratic function approximations ($p=2$) with $i=1$. Hence $\mathbf{n}^{(1)}=(0, 2)$ and

$$N^{(1)}=(0+1)+(2+1)-1=3=(p-i)q+ir.$$

The unique 1-multiplier of type $\mathbf{n}^{(1)}$ is given by

$$\mathbf{a}^{(1)}(z)=(0, (1-z)(1-2z))=(a_0(z), a_2(z)),$$

and hence we have by definition

$$a_1(z)=-e^z(1-3z).$$

The $(0, n_1, 2)$ algebraic form for $f(z)$ satisfies

$$a_{0,n_1}(z)+a_{1,n_1}(z)f(z)+a_{2,n_1}(z)f(z)^2=O(z^N)=O(z^{n_1+4}).$$

For computational convenience in this example, we have chosen to normalize this algebraic form so that $a_{2,n_1}(z)$ is a monic polynomial. Thus

$$a_{2,n_1}(z)=\beta+\alpha z+z^2,$$

$$a_{0,n_1}(z)=\gamma.$$

TABLE I

n_1	α	β	γ
0	-0.3061	0.7143×10^{-1}	0.2347
1	-0.9317	0.4144	0.2213
2	-0.9999	0.3816	0.1277
3	-1.3529	0.4314	0.1198×10^{-1}
4	-1.4504	0.4777	0.1091×10^{-2}
5	-1.4892	0.4949	0.7746×10^{-4}
6	-1.4981	0.4991	0.4522×10^{-5}
7	-1.4997	0.4999	0.2235×10^{-6}

The relation (2.11) implies that as $n_1 \to \infty$,

$$a_{2,n_1}(z) \to a_2(z) = \tfrac{1}{2} - \tfrac{3}{2}z + z^2 \qquad \text{(normalized)},$$
$$a_{0,n_1}(z) \to a_0(z) = 0.$$

Some values (rounded to the number of digits shown) of α, β, and γ for increasing n_1, are tabulated in Table I. They appear to be converging satisfactorily to the limiting values of -1.5, 0.5, and 0, respectively.

The relation (2.11) also implies that as $n_1 \to \infty$,

$$a_{1,n_1}(z) \to a_1(z) = -\tfrac{1}{2}e^z(1-3z). \qquad \text{(normalized)}$$

For $n_1 = 7$ in the present example we have

$$a_{1,7}(z) = -0.4999 + 0.9999z + 1.2495z^2 + 0.6660z^3 + 0.2285z^4 + 0.05765z^5 + 0.01119z^6 + 0.001496z^7.$$

This may be compared to

$$\begin{aligned} a_1(z) &= \tfrac{1}{2}e^z(1-3z) \\ &= -\tfrac{1}{2} + z + \tfrac{5}{4}z^2 + \tfrac{2}{3}z^3 + \tfrac{11}{48}z^4 + \tfrac{7}{120}z^5 + \tfrac{17}{1440}z^6 + \tfrac{1}{504}z^7 + \cdots \\ &= -0.5000 + 1.0000z + 1.2500z^2 + 0.6667z^3 \\ &\quad + 0.2292z^4 + 0.05833z^5 + 0.01181z^6 + 0.001984z^7 \\ &\quad + \cdots \qquad \text{(calculated to 4 significant figures)}. \end{aligned}$$

The "limiting" form is

$$P_\infty(f, z) = 0 - \tfrac{1}{2}(1-3z)\, e^z f(z) + \tfrac{1}{2}(1-z)(1-2z)\, f(z)^2,$$

and

$$\partial P_\infty(f, z)/\partial f = -\tfrac{1}{2}(1-3z)\, e^z + (1-z)(1-2z)\, f(z),$$

which is zero only at $z = \frac{1}{3}$. Note that as $n_1 \to \infty$, the discriminant of $Q_{n_1}(z)$ approaches $a_1(z)^2$, and has two branch points near $z = \frac{1}{3}$, which coalesce in the limit. Part (iii) of Theorem 2.3 implies that in a region excluding the points $z = \frac{1}{3}, \frac{1}{2}, 1$, the analytic continuation of the algebraic approximation $Q_{n_1}(z) \to f(z)$ as $n_1 \to \infty$.

In this example, using the polynomial coefficients calculated above, consider the branch

$$Q_7(z) = (-a_{1,7}(z) + \sqrt{a_{1,7}(z)^2 - 4a_{0,7}(z)\, a_{2,7}(z)})/(2a_{2,7}(z)),$$

taking the positive sign to satisfy the initial condition $Q_7(0) = 1 = f(0)$. If the exact values of the polynomial coefficients are taken (rather than the rounded approximations given above) then the relation $Q_7(z) = f(z) + O(z^{11})$ is obtained as expected.

References

1. G. A. Baker, Jr., "Essentials of Padé Approximants," Academic Press, New York, 1975.
2. G. A. Baker, Jr., and P. R. Graves-Morris, "Padé Approximants, Part I: Basic Theory, Part II: Extensions and Applications," Addison-Wesley, Reading, MA, 1981.
3. G. A. Baker, Jr., and D. S. Lubinsky, Convergence theorems for rows of differential and algebraic Hermite–Padé approximations, *J. Comput. Appl. Math.* **18** (1987), 29–52.
4. R. G. Brookes and A. W. McInnes, The existence and local behaviour of the quadratic function approximation, *J. Approx. Theory* **62** (1990), 383–395.
5. A. W. McInnes, Existence and uniquenes of algebraic function approximations, *Constr. Approx.* **8** (1992), 1–21.